Meet-irreducible elements in the poset of all logics

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What is a **Leibniz condition**?

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Jansana & Moraschini, *The poset of all logics I, II, III* (2021, 2023)

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The class of all Suszko models of $\vdash$ is denoted as $\mathrm{Mod}^{\equiv}(\vdash)$. These will be the 'intended models' of the logic $\vdash$.

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Given two algebraic languages $\mathcal{L}$ and $\mathcal{L}'$, a *translation of* $\mathcal{L}$ *into* $\mathcal{L}'$ is a map τ that sends each *n*-ary basic operation $*$ in $\mathcal{L}$ to an *n*-ary formula $\tau(*)$ in $\mathcal{L}'$ and in variables $x_1, \ldots, x_n$.

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Given an $\mathcal{L}'$-algebra $\mathcal{A}$, we define the *translation of* $\mathcal{A}$ *by* τ as the $\mathcal{L}$-algebra $\mathcal{A}^\tau$ with universe A and in which each n-ary operation $*$ of $\mathcal{L}$ is interpreted as follows:

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We can extend this interpretation for $\mathcal{L}$-terms in general (structural induction).

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Given a logic $\vdash$, denote by $\mathcal{L}(\vdash)$ the language in which it is formulated and by $\mathrm{Fm}(\vdash)$ the set of terms in $\mathcal{L}(\vdash)$.

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Let $\vdash, \vdash'$ be two logics. An *interpretation from* $\vdash$ *into* $\vdash'$ is a translation τ of $\mathcal{L}(\vdash)$ into $\mathcal{L}(\vdash')$ that preserves the relevant semantic information, i.e. such that

$$(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash') \text{ implies that } (\mathcal{A}^{\tau}, F) \in \mathrm{Mod}^{\equiv}(\vdash).$$

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We say that $\vdash$ is *interpretable into* $\vdash'$ if there is an interpretation from $\vdash$ into $\vdash'$. We write $\vdash \leqslant \vdash'$.

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The *'poset'* Log *of all logics* is the class of all the $[\![\vdash]\!]$ ordered as follows:

$$[\![\vdash]\!] \leqslant [\![\vdash']\!] \text{ if and only if } \vdash \leqslant \vdash' .$$

Non-indexed products

Non-indexed products

Given a family of algebraic languages $\{\mathcal{L}_i \mid i \in I\}$, we define the language $\bigotimes_{i \in I} \mathcal{L}_i$ whose n-ary basic operations are defined as:

$$(\varphi_i(x_1, \dots, x_n))_{i \in I},$$

where $\varphi_i(\overline{x})$ is an n-ary $\mathcal{L}_i$-term, for each $i \in I$.

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That is, we take a language whose n-ary function symbols are tuples of n-ary terms in each of the initial languages.

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- Its universe is $\prod_{i\in I} A_i$.

- Each n-ary basic operation $(\varphi_i(x_1, \dots, x_n))_{i\in I}$ of $\bigotimes_{i\in I} \mathcal{L}_i$ is interpreted as follows:

$$(\varphi_i(x_1, \dots, x_n))_{i\in I}^{\bigotimes_{i\in I} \mathcal{A}_i}(\overline{a}_1, \dots, \overline{a}_n) := (\varphi_i^{\mathcal{A}_i}(\overline{a}_1(i), \dots, \overline{a}_n(i)))_{i\in I},$$

where $\overline{a}_1, \dots, \overline{a}_n \in \prod_{i\in I} A_i$.

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Given a family of matrices $\{(\mathcal{A}_i, F_i) \mid i \in I\}$ in which each $\mathcal{A}_i$ is a $\mathcal{L}_i$-algebra, we can similarly define the $\bigotimes_{i \in I} \mathcal{L}_i$-matrix

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Moreover, if for each $i \in I$ we have a class $\mathbb{K}_i$ of $\mathcal{L}_i$-matrices, we define the class:

$$\bigotimes_{i \in I} \mathbb{K}_i := \mathbb{I} \left\{ \bigotimes_{i \in I} (\mathcal{A}_i, F_i) \mid (\mathcal{A}_i, F_i) \in \mathbb{K}_i, i \in I \right\}.$$

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Let $\{\vdash_i \mid i \in I\}$ be a family of logics. We define the *non-indexed product* of such family as the logic $\bigotimes_{i \in I} \vdash_i$ in the language $\bigotimes_{i \in I} \mathcal{L}(\vdash_i)$ with $\prod_{i \in I} |\mathrm{Fm}(\vdash_i)|$ variables and induced by the class $\bigotimes_{i \in I} \mathrm{Mod}^{\equiv}(\vdash_i)$.

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Two main reasons:

- It behaves well with respect to Suszko semantics.
- It corresponds to a structural feature of the poset Log.

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Theorem (Jansana & Moraschini)

$\mathrm{Mod}^{\equiv}\left(\bigotimes_{i\in I} \vdash_i\right) = \mathbb{P}_{\mathrm{SD}}\left(\bigotimes_{i\in I} \mathrm{Mod}^{\equiv}(\vdash_i)\right).$

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The binary case works even more smoothly:

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Moreover:

Theorem (Jansana & Moraschini)

*The infimum of a sub***set** $\{[\![\vdash_i]\!] \mid i\in I\}$ *of* Log *is precisely* $[\![\bigotimes_{i\in I}\vdash_i]\!]$.

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Nevertheless, we are now able to talk about meet-irreducibility in Log.

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We say that a logic $\vdash$ is *meet-irreducible* when $[\![\vdash]\!]$ is a meet-irreducible element of Log, i.e., for every pair of logics $\vdash_1$ and $\vdash_2$,

$$[\![\vdash_1 \otimes \vdash_2]\!] = [\![\vdash]\!] \text{ implies that either } \vdash_1 \leqslant \vdash \text{ or } \vdash_2 \leqslant \vdash .$$

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The idea is that $\vdash$ cannot be recovered as the non-indexed product of any pair of logics, none of which is equi-interpretable to it. In this sense, it is **fundamental**.

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In a similar way, we say that a logic $\vdash$ is *meet-prime* when, for every pair of logics $\vdash_1$ and $\vdash_2$,

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A logic $\vdash$ *has irreducibility degree* $\leqslant k$ if, given $\vdash_0, \dots, \vdash_k$,

$[\![\vdash_0]\!] \wedge \cdots \wedge [\![\vdash_k]\!] = [\![\vdash]\!]$ implies that $[\![\vdash_0]\!] \wedge \cdots \wedge \cancel{[\![\vdash_i]\!]} \wedge \cdots \wedge [\![\vdash_k]\!] \leqslant [\![\vdash]\!]$,

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A logic $\vdash$ *has Helly number* $\leqslant k$ if, given $\vdash_0, \dots, \vdash_k$,

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for some $i \leqslant k$.

Note that we recover the previous definitions for $k = 1$.

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A class $\mathbb{K}$ of matrices has the *joint embedding property* (JEP) if for every set X of $\mathbb{K}$, whose members are all different from $(\mathbf{1}, \varnothing)$, there is an element of $\mathbb{K}$ in which every member of X embeds.

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Theorem (1)

Every logic with theorems $\vdash$ satisfying the following conditions has irreducibility degree $\leqslant k$ in Log:

(1) $\mathrm{Mod}^{\equiv}(\vdash)$ *has the JEP.*

(2) Every nontrivial element of $\mathrm{Mod}^{\equiv}(\vdash)$ *has a substructure of cardinality* $p_1 \cdots p_t$ *with* $t \leqslant k$ *and each* p_i *prime.*

(3) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ *do not have trivial substructures.*

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For $k = 1$ we immediately have:

Corollary

Every logic with theorems $\vdash$ satisfying the following conditions is meet-irreducible in Log:

(1) $\mathrm{Mod}^{\equiv}(\vdash)$ has the JEP.

(2) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ have substructures of prime cardinality.

(3) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ do not have trivial substructures.

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Let L be a sub-semilattice of Log and let $\vdash$ be such that $[\![\vdash]\!] \in \mathrm{L}$. Then, we say that $\vdash$ is meet-prime (resp. irreducible) *in* L if the meet-primeness (resp. irreducibility) condition holds for arbitrary logics $\vdash_1, \vdash_2$ such that $[\![\vdash_1]\!], [\![\vdash_2]\!] \in \mathrm{L}$.

Meet-irreducibility: more criteria

We can restrict ourselves to certain sub-semilattices of Log.

Let L be a sub-semilattice of Log and let $\vdash$ be such that $[\![\vdash]\!] \in \mathrm{L}$. Then, we say that $\vdash$ is meet-prime (resp. irreducible) *in* L if the meet-primeness (resp. irreducibility) condition holds for arbitrary logics $\vdash_1, \vdash_2$ such that $[\![\vdash_1]\!], [\![\vdash_2]\!] \in \mathrm{L}$.

The same applies for having Helly number (resp. irreducibility degree) below *k in* L.

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Question: can we weaken the meet-irreducibility condition of Theorem 1 into that of meet-primeness?

We say that a logic $\vdash$ is *hereditarily nontrivial* if nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ do not have trivial substructures.

Consider the filter HNT^{*} of Log given by the collection of all the $[\![\vdash]\!]$, where $\vdash$ is a hereditarily nontrivial logic with theorems.

Meet-irreducibility: more criteria

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Theorem (2)

Every logic (with theorems) $\vdash$ *satisfying the following conditions has Helly number* $\leqslant k$ *in* HNT^**:*

1. $\mathrm{Mod}^{\equiv}(\vdash)$ *has the JEP.*
2. *Every nontrivial element of* $\mathrm{Mod}^{\equiv}(\vdash)$ *has a substructure of cardinality* $p_1 \cdots p_t$ *with* $t \leqslant k$ *and each* p_i *prime.*
3. *Nontrivial elements of* $\mathrm{Mod}^{\equiv}(\vdash)$ *do not have trivial substructures.*

Meet-irreducibility: more criteria

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Again, for $k = 1$:

Corollary

Every logic (with theorems) $\vdash$ satisfying the following conditions is meet-prime in HNT^*:

(1) $\mathrm{Mod}^{\equiv}(\vdash)$ has the JEP.
(2) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ have substructures of prime cardinality.
(3) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ do not have trivial substructures.

Meet-irreducibility: more criteria

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We say that a logic $\vdash$ can be written as a *nested intersection* if there is a family $\{\vdash_i \mid i \in \omega\}$ of logics in the language $\mathcal{L}(\vdash)$ such that $\vdash_i \subseteq \vdash_j$ if $j \leqslant i$ and $\vdash = \bigcap_{i \in \omega} \vdash_i$.

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Question: can we give a criterion for nested intersections of logics?

Meet irreducibility: more criteria

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Recall that a logic $\vdash$ is *finitely equivalential* if it is finitary and there is a finite and **nonempty** set of formulas $\Delta(x, y)$ such that:

$$\emptyset \vdash \Delta(x, x); \qquad x, \Delta(x, y) \vdash y;$$

$$\bigcup_{i \leqslant n} \Delta(x_i, y_i) \vdash \Delta(*(\overline{x}), *(\overline{y})),$$

for every n-ary basic operation $*$ of $\mathcal{L}(\vdash)$.

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for every n-ary basic operation $*$ of $\mathcal{L}(\vdash)$.

We define the filter FinEq of Log given by the collection of all the $[\![\vdash]\!]$, where $\vdash$ is a finitely equivalential logic (with theorems).

Meet-irreducibility: more criteria

Meet-irreducibility: more criteria

Since every finitely equivalential logic $\vdash$ has theorems:

Corollary

Every finitely equivalential logic $\vdash$ satisfying the following conditions is meet-irreducible in FinEq:

(1) $\mathrm{Mod}^{\equiv}(\vdash)$ has the JEP.

(2) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ have substructures of prime cardinality.

(3) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ do not have trivial substructures.

Meet-irreducibility: more criteria

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Theorem (3)

Let $\{\vdash_n \mid n \in \omega\}$ be a collection of finitely equivalential logics with theorems such that $\vdash_n$ is an extension of $\vdash_m$ if $n \leqslant m$. Moreover, assume that each of these logics verifies the conditions (1), (2) and (3) from Theorem 1.
Then, if $\bigcap_{n\in\omega} \vdash_n$ is a finitely equivalential logic, it is in fact a meet-irreducible element in $\mathrm{FinEq} \cap \mathrm{HNT}^*$.

Applications

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We can verify the conditions of Theorem 1 for concrete examples of logics.

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Theorem

The following logics are meet-irreducible in Log:

(i) *The intermediate logics.*

(ii) *The global consequence logics for the normal modal logics* $S4$ *and* $S5$.

Applications

Theorem

Łukasiewicz logic is meet-irreducible in Log.

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Since $\vdash_{Ł}$ has theorems, it is enough to check conditions (1), (2) and (3) from Theorem 1.

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Recall that, since $\vdash_{\text{Ł}}$ is algebraized by $\mathbf{MV}$, $\Delta(x, y) := \{x \to y, y \to x\}$ and $\tau(x) := \{x \approx 1\}$,

$$\mathrm{Mod}^{\equiv}(\vdash_{\text{Ł}}) = \{(\mathcal{A}, \tau(\mathcal{A})) \mid \mathcal{A} \in \mathbf{MV}\} = \{(\mathcal{A}, \{1\}) \mid \mathcal{A} \in \mathbf{MV}\}.$$

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Recall **Chang's theorem**: $\mathbf{MV} = \mathbf{Q}([0, 1]_{Ł})$.

Applications

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Given a class $\mathbb{K}$ of $\mathcal{L}$-matrices, we can regard each of its elements as a $\mathcal{L}_R$-structure, where $\mathcal{L}_R := (\mathcal{L}, \{R\})$ and R is a unary relation symbol interpreted in each $(\mathcal{A}, F) \in \mathbb{K}$ as F.

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Recall **Maltsev theorem**: a $\mathcal{L}_R$-quasivariety $\mathbb{K}$ has the JEP if and only if there is some $\mathcal{L}_R$-structure $\mathfrak{A} \in \mathbb{K}$ such that $\mathbb{K} = \mathbf{Q}_R(\mathfrak{A})$, i.e. $\mathbb{K}$ is the least $\mathcal{L}_R$-quasivariety containing $\mathfrak{A}$.

Applications

Proof.

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By Maltsev theorem, it is enough to find $(\mathcal{A}, \{1\}) \in \mathrm{Mod}^{\equiv}(\vdash_{Ł})$ such that $\mathrm{Mod}^{\equiv}(\vdash_{Ł}) = \mathbf{Q}_R((\mathcal{A}, \{1\}))$.

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By Chang's theorem,

$$\mathrm{Mod}^{\equiv}(\vdash_{Ł}) = \{(\mathcal{A}, \{1\}) \mid \mathcal{A} \in \mathbf{Q}([0,1]_{Ł})\} =$$

$$= \{(\mathcal{A}, \{1\}) \mid \mathcal{A} \in \mathbb{ISPP}_{\mathrm{U}}([0,1]_{Ł})\} = \mathbf{Q}_R([0,1]_{Ł}).$$

$\square$

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- Recall that the two-element chain $Ł_2$ can be embedded into every nontrivial element of **MV**.
- Given a nontrivial element $(\mathcal{A}, \{1\}) \in \mathrm{Mod}^{\equiv}(\vdash_{Ł})$, we know that $Ł_2 \hookrightarrow \mathcal{A}$.

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Recall that the two-element chain Ł_2 can be embedded into every nontrivial element of **MV**.

Given a nontrivial element $(\mathcal{A}, \{1\}) \in \mathrm{Mod}^{\equiv}(\vdash_{\text{Ł}})$, we know that $\text{Ł}_2 \hookrightarrow \mathcal{A}$.

Hence, $(\text{Ł}_2, \{1\}) \hookrightarrow (\mathcal{A}, \{1\})$.

$\square$

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(3) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash_{\text{Ł}})$ do not have trivial substructures.

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Every submatrix of a nontrivial element of $\mathrm{Mod}^{\equiv}(\vdash_{Ł})$ will contain $(Ł_2, \{1\})$ as a submatrix by closing under interpretation of constants $(1 := \neg 0)$.

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Hence, it will be nontrivial. $\square$

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We have seen how some useful ideas from universal algebra can be applied to the analysis of the poset of all logics.

Some questions remain:

- Are the presented results improvable?
- Can more criteria for meet-irreducibility in Log be worked out?
- Which is the philosophical relevance of these results?

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