Indicative conditionals: some algebraic considerations

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Non-example: *if Lee Harvey-Oswald had not shot JFK, someone else would have.*

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Non-example: *if Lee Harvey-Oswald had not shot JFK, someone else would have.*

Hence, one ought to distinguish between indicative conditionals and counterfactuals.

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- Cooper's ordinary language logic,
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(iii) and it is called off when ϕ is not realized.

It verifies *Adams thesis*:

$$\mathbf{p}(\varphi \to \psi \text{ is true} \mid \varphi \to \psi \text{ has classical value}) = \mathbf{p}(\psi \mid \varphi).$$

Reichenbach-De Finetti logic

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Consider the algebra $\mathbf{DF}_3$ of universe $\{\mathbf{0}, {}^1\!/\!_2, \mathbf{1}\}$ and operations:

	$\neg$
$\mathbf{0}$	$\mathbf{1}$
$1/2$	$1/2$
$\mathbf{1}$	$\mathbf{0}$

$\wedge_K$	$\mathbf{0}$	$1/2$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$1/2$	$\mathbf{0}$	$1/2$	$1/2$
$\mathbf{1}$	$\mathbf{0}$	$1/2$	$\mathbf{1}$

$\vee_K$	$\mathbf{0}$	$1/2$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$1/2$	$\mathbf{1}$
$1/2$	$1/2$	$1/2$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

$\rightarrow_{DF}$	$\mathbf{0}$	$1/2$	$\mathbf{1}$
$\mathbf{0}$	$1/2$	$1/2$	$1/2$
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$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
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$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

$\to_{\mathsf{DF}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$
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Interestingly, we can interchange the language with $\{\neg, \wedge_{\mathsf{K}}, \vee_{\mathsf{K}}, \mathbf{1/2}\}$ by letting

$$x \to_{\mathsf{DF}} y := (\mathbf{1/2} \wedge_{\mathsf{K}} \neg x) \vee_{\mathsf{K}} (x \wedge_{\mathsf{K}} y)$$

and, conversely,

$$\mathbf{1/2} := \neg x \to_{\mathsf{DF}} (x \to_{\mathsf{DF}} x).$$

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- $\mathrm{DF}^{\mathbf{1}} := \mathrm{Log}(\mathbf{DF}_3, \{\mathbf{1}\})$ is an expansion of strong Kleene's logic.

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- DF can be axiomatized by adding the axioms

$$\vdash \mathbf{1/2} \text{ and } \vdash \neg\mathbf{1/2}$$

  to any complete calculus $\vdash$ for Priest's logic.

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- Similar comments apply for $\mathrm{DF}^{\wedge_K} := \mathrm{Log}[(\mathbf{DF}_3, \{\mathbf{1/2}, \mathbf{1}\}), (\mathbf{DF}_3, \{\mathbf{1}\})]$ and $\mathrm{DF}^{\mathbf{1}}$.

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(i) the former sanctions intuitively false implications:

2.10 *If Goldbach's conjecture is correct, then it is false that if the mayor's telephone number is an even number, it cannot be represented as the sum of two primes. Therefore, if the mayor's telephone number is not an even number, Goldbach's conjecture is not correct.*

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$G \to \neg(E \to P) \sim (E \to P) \to \neg G \sim (\neg E \vee P) \to \neg G.$

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(ii) the latter sanctions classically false implications:

2.19 *If the water supply is off then she cannot cook dinner, or if the gas supply is off then she cannot cook dinner. Therefore, if either the water supply or the gas supply is off, then she cannot cook dinner.*

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$(W \to C) \vee (G \to C) \nsim (W \vee G) \to C.$

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$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1}$	$\mathbf{0}$

$\wedge_{\mathsf{OL}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{1/2}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$

$\vee_{\mathsf{OL}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$
$\mathbf{1/2}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
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$\to_{\mathsf{OL}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$
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Originally, atomic propositions could not be evaluated as $^1\!/_2$... but the resulting logic OL is nonstructural!

Cooper's ordinary language logic

Consider the algebra $\mathbf{OL}_3$ of universe $\{\mathbf{0}, \mathbf{1/2}, \mathbf{1}\}$ and operations:

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$\mathbf{1}$	$\mathbf{0}$

$\wedge_{\mathrm{OL}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
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Let $\mathrm{sOL} := \mathrm{Log}(\mathbf{O}_3, \{\mathbf{1/2}, \mathbf{1}\})$.

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Let $\mathrm{sOL} := \mathrm{Log}(\mathbf{O}_3, \{\mathbf{1/2}, \mathbf{1}\})$. Then, $\mathrm{OL} = \mathrm{sOL} + \varphi \wedge \neg\varphi \vdash \psi$.

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- sOL can be axiomatized by extending the classical $\{\wedge, \vee\}$-fragment with axioms
    - i. $\neg\neg\varphi \leftrightarrow \varphi$,
    - ii. $(\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi$,
    - iii. $\neg(\varphi \wedge \psi) \leftrightarrow ((\varphi \rightarrow \neg\psi) \wedge (\psi \rightarrow \neg\varphi))$,
    - iv. $\neg(\varphi \rightarrow \psi) \leftrightarrow (\varphi \rightarrow \neg\psi)$ (*Boethius theses*).

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- De Finetti's logic is too a subsystem of sOL, since $\mathbf{1/2}$ is definable in $\{\neg, \to_{\mathsf{OL}}\}$.

Farrell-Cantwell logic

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$\mathbf{F}_3$: $\neg, \wedge_\mathsf{K}, \vee_\mathsf{K}$ with

$\to_\mathsf{F}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1/2}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1/2}$
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(b) Cantwell's motivations: interpret the previously considered three-valued negation as a 'conditional negation'. Then, $\neg\varphi$ is interpreted as 'φ is false if it has (classical) truth-value'.

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$\mathbf{CN}_3$: $\neg, \wedge_K, \vee_K$ with

$\to_{OL}$	$\mathbf{0}$	$1/2$	$\mathbf{1}$
$\mathbf{0}$	$1/2$	$1/2$	$1/2$
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$$\mathrm{F} := \mathrm{Log}(\mathbf{F}_3, \{\mathbf{1/2}, \mathbf{1}\}) \text{ and } \mathrm{CN} := \mathrm{Log}(\mathbf{CN}_3, \{\mathbf{1/2}, \mathbf{1}\})$$

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Then:

– Both logics are definitionally equivalent letting

$$x \to_{\mathsf{OL}} y := \neg((y \to_{\mathsf{F}} x) \to_{\mathsf{F}} \neg y) \to_{\mathsf{F}} ((x \vee_{\mathsf{K}} y) \to_{\mathsf{F}} y),$$

and

$$x \to_{\mathsf{F}} y := x \to_{\mathsf{OL}} (x \wedge_{\mathsf{K}} y).$$

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– Both logics are algebraizable and axiomatizable.

Farrell-Cantwell logic

Let

$$\mathrm{F} := \mathrm{Log}(\mathbf{F}_3, \{\mathbf{1/2}, \mathbf{1}\}) \text{ and } \mathrm{CN} := \mathrm{Log}(\mathbf{CN}_3, \{\mathbf{1/2}, \mathbf{1}\})$$

Then:

- Both logics are definitionally equivalent letting

  $$x \to_{\mathsf{OL}} y := \neg((y \to_{\mathsf{F}} x) \to_{\mathsf{F}} \neg y) \to_{\mathsf{F}} ((x \vee_{\mathsf{K}} y) \to_{\mathsf{F}} y),$$

  and

  $$x \to_{\mathsf{F}} y := x \to_{\mathsf{OL}} (x \wedge_{\mathsf{K}} y).$$

- Both logics are algebraizable and axiomatizable.
- Both logics are subsystems of sOL.

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iv. Get back to the philosophical motivations of these logics.

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